# THE INTERFACE OF QUANTIFICATION AND COVARIATIONAL REASONING IN REAL WORLD SCENARIOS

Jennifer Czocher
Texas State University
Czocher.1@txstate.edu

Elizabeth Roan
Texas State University
Eaw109@txstate.edu

Carlos Acevedo
Texas State University
Cia24@txstate.edu

Micah Swartz
Texas State University
Mas762@txstate.edu

*Quantitative and covariational reasoning (QCR) are foundational to productive conceptualizations of functions, and especially to properties belonging to first and second derivatives. Through the lens of QCR, we explore how derivatives and rates of change might influence mathematical model construction. Drawing on sessions from an exploratory teaching experiment with an undergraduate STEM major, we illustrate reasoning with non-normative conceptions of derivatives that is consistent and robust and conjecture how this reasoning impacts model construction.*

Keywords: modeling, calculus, mathematical representations, undergraduate mathematics

Many STEM disciplines rely on mathematical models to convey meaning. The kinds of models studied in advanced mathematics and STEM coursework regularly feature first and second derivatives, sometimes in relation to a single independent variable and sometimes in relations among themselves. Thus, relational properties among these quantities like dependence, directionality, and coordination-of-change take on additional importance when considering students' meanings for the models that recruit these quantities. It is yet to be understood how the manner of quantification of specific quantities may constrain covariational reasoning and subsequently the kinds of scenario-based conditions modelers may carry forward into their models. The purpose of this paper is to examine the interface of quantification and covariational reasoning about the first and second derivative.

## Theoretical Perspective and Empirical Background

Researchers have elaborated theoretical constructs such as quantitative reasoning and covariational reasoning (QCR) for explaining, predicting, and extending students' thinking in the presence of variation (Thompson & Carlson, 2017) while at the same time, studies of mathematical modeling processes have been incorporating methods capable of accounting for modelers' QCR (e.g., Czocher & Hardison, 2021). By *mathematical model*, we mean a conceptual system accessible through the modeler's mathematical expression of locally meaningful representational systems for real-world phenomena. Coordinating two varying quantities and attending to relationships among them is *covariational reasoning* (Carlson et al., 2002). It presents as patterns of reasoning that compares quantities, combine them through operations, trace their changes, rates of changes, and intensities of changes (Johnson, 2015). According to Carlson et al. (2002), covariational reasoning passes through five levels of development based in the individual's imagery of the dynamics and relative to a task scenario. Each level corresponds to increasingly sophisticated mental actions while retaining the nature of mental actions associated with all lower levels: MA1 – *dependence* of one variable on another, MA2 – *direction* of change of one variable with changes in the other, MA3 – *amount* of change of one variable with changes in the other, MA4 – *average* rate-of-change of one variable with uniform increments of the other, MA5 – *instantaneous* rate-of-change of one variable with continuous changes in the other (Carlson et al., 2002). Jones (2016) studied students' conceptions of second derivative and concavity, elaborating on the covariational reasoning levels. He argued that, if one considers rate-of-change in a quantity as the dependent variable, then reasoning about concavity can be recast as mental actions MA4_1 (dependence of rate of change on independent variable), MA4_2 (direction of change in the rate of change with respect to the independent variable), and MA4_3 (amount of change in the rate of change with changes in independent variable). Conceptualizing mental actions 1, 2, and 3 applied to variation of rate-of-change along with mental actions 4 and 5 applied to variation in the base variable foreshadows ways of thinking reported in by Jones (2019), where participants conflated the magnitude of rate of change with its directionality.

Taken together, the literature points toward students' quantification as an explanatory mechanism for their modeling activities and especially toward the quantitative and covariational relationships students formulate as a basis for the graphical or symbolic expressions they create to communicate those relationships. Our methodology, described below, is borne out from these considerations as we seek to understand how an individual conceives of relations among time, a quantity, and a rate-of-change of that quantity and the models occasioned by those relations. We address the question: How does quantification of the first and second derivative influence covariational reasoning and what might be its collective impact on model construction?

## Methods

Data comes from two task-based interview sessions drawn from exploratory teaching experiment with an undergraduate STEM major, Azure, focused on uncovering how to leverage and extend students' quantitative reasoning for the purpose of creating and expressing mathematical models of real-world scenarios. The sessions treat the Ice Melt Task, which presents a set of scenarios where ice is placed in contrasting environments. The participants are asked to distinguish among magnitude and sign of volume and rate of change of volume. Follow up questioning occasioned consideration of pairwise covariational comparisons of time, quantity, and rate of change of quantity, and to communicate properties of those conceived relationships through graphs and symbolic representations. Data analysis first sought instances of Azure's reasoning consistent with Jones's (2016, 2019) conceptual descriptions of concavity and where he made comparisons to physics concepts like velocity. We then catalogued the situationally-relevant quantities Azure imputed to the scenario, applying Czocher & Hardison's (2021) quantification criteria. Finally, we analyzed the instances identified in the first pass by examining the mental actions (MA1-MA5; MA4_1-MA4_3) and levels of covariation (CL1-CL5) the relevant pairs of quantities (identified in the second pass) permitted (Carlson et al., 2002; Jones, 2016).

## Results

In total, Azure imputed 5 quantities with situational references relevant to the research question: Volume (amount of ice), Ambient Temperature (temperature of environment surrounding ice), Rate of Change of Volume (absolute change in volume between two distinct times; ice), Rate of Change (of Rate of Change) of Volume (rate at which ice melting), and Time (the indefinite continued progress of existence and events). Additionally, Azure quantified *slope* without a situational referent that nevertheless had a meaningful referent in his graphing activity. We discuss his non-normative situational referents for the rates of change below.

Azure described volume of ice in different scenarios in interesting ways, for example an iceberg in the ocean has a small/positive volume while ice in a cup has large/positive volume. He demonstrated coordination of $V$, volume of water, and time $t$ through mental action (MA) 1 and covariational reasoning level (CL) 1. For example, he drew the graph in Figure 1, which along with his explanations suggested that as $t$ changes, so must $V$. He was aware that $V$ would decrease as time increased. In the next session, he stated that ice volume would decrease as time changed. In the tasks, he argued for a negative directionality between changes in $V$ across two points in time, evident in his graphs in Figure 1. Thus, he evidenced MA2 for volume and time.

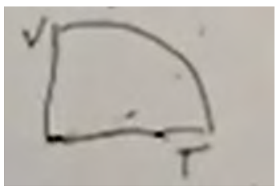


**Figure 1: V-t graphs for Ice Melt, first session (left) and second session (right)**

However, Azure did not extend his coordination of the direction of $V$ with $t$ in an anticipated way. Azure asserted that $dV/dt$ would be positive or zero for the scenario-based conditions of the Ice Melt Task, for example, an iceberg in the ocean was said to have a small/positive $dV/dt$, and though he established that the direction of change of $V$ with respect to $t$ depended on ambient temperature, he did not evidence thinking that the rate-of-change of $V$ with respect to $t$ would change sign dependent upon ambient temperature. We interpret he coordinated $dV/dt$ and $t$ through MA1, MA3, MA4 and CL1, CL4. Azure did successfully and consistently coordinate direction of change of the magnitude of $dV/dt$, and so it is unclear whether 'credit' for MA2 should be given, according to the covariational reasoning framework. Specifically, he sketched Figure 2 (left) to represent an ice cube dropped into a hot cup of coffee. His figure shows $dV/dt$ above the $t$-axis, with a positive sign, but he stated that the rate-of-change of volume was decreasing. He explained, "the slope of volume versus time graph is the magnitude of the $dV/dt$", suggesting that he associated slope with the directional coordination of $V$ and $t$ (MA2) but that he associated magnitude of $dV/dt$, which is always positive, with the rate that $V$ changes with respect to $t$ (also MA2, but for magnitude). Further complicating Azure's covariational reasoning, and our interpretations of it, was his quantification of rate-of-change of volume with respect to time. He defined rate-of-change of volume with respect to time as "the comparison [of volume] between two different points in time." Thus, he conceived rate-of-change as a displacement, an always-positive quantity. This would offer some confirmatory support to him when checking his own reasoning about the sign of $dV/dt$, or at the least, would not be a source of cognitive conflict in his reasoning with $dV/dt$.

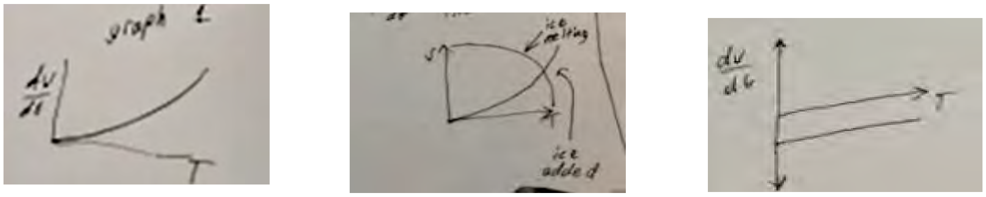


**Figure 2: dV/dt-t graph for Ice Melt first session (left), V-t graphs for ice melting and ice added (middle), dV/dt-t graph for Ice Melt second session (right).**

Azure evidenced imagery of amount-of-change of volume and rate-of-change of volume changing with respect to time. He stated that the rate-of-change of volume would be positive and increasing with respect to time. Azure drew two curves on the same $V$-$t$ axes (Figure 2, middle) and stated, "either one of these, depending on how the problem is worded" would be correct, which suggests, similar to Jones (2019) argument, that the different quantifications of derivative can clash. Because Azure conceived of rate-of-change as always positive, the same $dV/dt$-$t$ graph could represent either increase or decrease in volume. He explained that rate-of-change being positive either means that the volume is increasing (so ice is being added) or volume is decreasing (ice is melting). He noted there was no way to tell from the graph which scenario was modeled; he would need more information about the ambient temperature of the room and if water were available to re-freeze.

Azure supplied evidence that he could coordinate the amount of change of rate-of-change with change-in-time. In one instance, he argued "if it [$dV/dt$] is horizontal, regardless of if it's above or below the $x$-axis [$t$-axis], it [volume] is changing. But if the $V$-$t$ graph is anything but a straight line, if it's one of these lines (indicating Figure 2 middle), the steeper this curve gets, the more $dV/dt$-$t$ graph is a straight line up or down." Working from Figure 1 (right) and Figure 2 (right), he appealed to a quantification of steepness of the graph (here the "situational referent" is figurative material in the graph). He referred to the steepness property of the $V$-$t$ graph as an indicator of how closely $dV/dt$-$t$ graph should resemble a vertical line. These latter instances are indicative of Jones' extended MA4_1 and MA4_3. Azure attended to multiple attributes of volume, so it was difficult to clearly attribute MA4 and MA5 to his reasoning. He conceived negative rate-of-change of volume as equivalent to positive rate-of-change of volume when absolute "change between the initial state and the final state" are the same.

## Discussion and Conclusions

Azure adeptly coordinated both change and change-in-change with time and was able to coordinate change and change-in-time with one another, by appealing to graphical properties, real-world reasoning, and without evidencing MA4 and MA5 for the $V$-$t$ covariation. This observation supports Jones (2019) arguments. Azure's conception of derivative was associated with multiple attributes of a situational referent. Sometimes $dV/dt$ meant absolute change in volume across two times and at others meant changing intensity of that change. His conception of instantaneous rate-of-change corresponded to imagery of steepness of slope of the $V$-$t$ graph, for an arbitrary time. Despite his non-normative conceptions and meanings for symbolic notation, his reasoning was consistent and correct when thinking through relations between change, rate-of-change, changes-in-change for volume and time, especially when illustrated graphically. However, his conceptions would be counterproductive for deriving models represented with arithmetic operations. Because he did not distinguish between freezing/melting graphically, he would not be able to use the graphs or his covariational reasoning to validate symbolic models and may come to inadequate conclusions about the validity of his models. We hypothesize that developing an (adequate to Azure) symbolic relationship as a model under these conditions would be challenging because of conditions he implicitly or explicitly imposed, like asymptotic behavior of the derivative directly caused by non-directionality covariation between the quantity of interest and time.

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